THEORY OF THERMAL CONDUCTION WHEN THE RATE OF HEAT PROPAGATION IS FINITE

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It is assumed that the rate of increase of entropy and of internal energy depend on the temperature and on the first partial derivative of the temperature with respect to the coordinates and time. This assumption enables us to obtain a heat-conduction equation of the hyperbolic type from the law of conservation of energy.

1. The equations which describe the propagation of heat in solids are given in [1-4 etc]. These equations have the form of wave equations. Relaxation of the heat flow is assumed, and this leads to a heat-conduction equation of the hyperbolic type. In the present paper we adopt a different approach. We assume that the solid medium is not deformed and is at rest. The heat-conduction equation will be obtained from the law of conservation of energy [5]:

$$dU/dt = -\operatorname{div}\mathbf{q} + \varepsilon. \qquad (1.1)$$

Equation (1.1) must be considered together with the second law of thermodynamics

$$\sigma = -\mathbf{q}\cdot\nabla T/T^2 \geqslant 0. \qquad (1.2)$$

To satisfy inequality (1.2) it is natural to assume that the heat flow q is in the opposite direction to the vector ∇T, i.e.,

$$\mathbf{q} = -\Lambda\nabla T. \qquad (1.3)$$

The function Λ will be called the heat conduction function or simply the heat conduction. If we substitute Eq. (1.3) into inequality (1.2) we obtain $\Lambda \leq 0$. In the simplest case of a linear Fourier heat conduction law ($\Lambda = \lambda$ = const) the heat conduction function is identical with the thermal conductivity. In the general case we will assume that Λ depends on temperature and on the first partial derivatives of the temperature with respect to the geometrical coordinates and time

$$\Lambda = \Lambda(T,\ \nabla T,\ \partial T/\partial t). \qquad (1.4)$$

The heat conduction Λ determines the amount of heat which is transmitted from the hotter part of the body to the cooler part as a result of which there is a redistribution of the temperature. Hence, when there are no sources of heat Λ causes a heating of the "cold" parts and a cooling of the "hot" parts of the body simultaneously. Since the sign of the partial derivative $\partial T/\partial t$ indicates the direction of the process (heating or cooling), it can be assumed that the function Λ is independent of the direction of this process, i.e., Λ should depend on $|\partial T/\partial t|$. If we assume that Λ is a scalar isotropic function of limited modulus, Eq. (1.4) can be rewritten in the form [6]

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$$\Lambda = \Lambda(T,\ N,\ |\partial T/\partial t|),\ N^2 = \nabla T \cdot \nabla T,\ \Lambda < \infty. \tag{1.5}$$

As far as the internal energy is concerned we will assume that it depends on T, N and $\partial T/\partial t$, i.e.,

$$U = U(T,\ N,\ \partial T/\partial t). \tag{1.6}$$

The considerations which lead us to assume that Λ depends on $|\partial T/\partial t|$, do not apply to the function U. Hence, Λ and U, in general, depend on different variables. It is necessary to impose the following limitations on the dependence of the function U on its arguments:

$$U = 0 \text{ for } T = 0,\ 0 \leqslant U < \infty,\ \partial U/\partial T = c > 0. \tag{1.7}$$

The quantity c is called the specific heat capacity. Substituting Eqs. (1.3) and (1.6) into Eq. (1.1) and taking Eq. (1.5) into account we obtain the heat conduction equation

$$\begin{gathered}\Lambda \Delta T + \frac{\partial \Lambda}{\partial T} N^2 + \frac{\partial \Lambda}{\partial N} \nabla N \cdot \nabla T + \frac{\partial \Lambda}{\partial |\partial T/\partial t|} \nabla T \cdot \nabla \left| \frac{\partial T}{\partial t} \right| + \\ + \varepsilon = c \frac{\partial T}{\partial t} + \frac{\partial U}{\partial N} \frac{\partial N}{\partial t} + \frac{\partial U}{\partial (\partial T/\partial t)} \frac{\partial^2 T}{\partial t^2}.\end{gathered} \tag{1.8}$$

Note that the assumptions regarding relations (1.5) and (1.6) lead to a heat-conduction equation (1.8) with higher derivatives of T of the second order. If we assume that the functions Λ and U depend not only on the temperature but on its first partial derivatives, and also on higher-order partial derivatives, we obtain a heat-conduction equation with a higher derivative of the temperature of greater than the second order. We will now study the properties of Eq. (1.8).

It is well known [7] that the characteristic surfaces coincide with the surfaces of weak first-order discontinuity, and propagate with the same velocity. Using the geometrical and kinematic relations on first-order discontinuities to obtain the velocity G of propagation of the characteristics of Eq. (1.8), we obtain the quadratic equation

$$\begin{gathered}aG^2 - 2bG - e = 0,\ a = \frac{\partial U}{\partial (\partial T/\partial t)}, \\ 2b = \frac{1}{N} \frac{\partial T}{\partial n} \frac{\partial U}{\partial N} - \frac{\partial \Lambda}{\partial |\partial T/\partial t|} \frac{\partial T}{\partial n} \operatorname{sign}\left(\frac{\partial T}{\partial t}\right),\ e = \Lambda + \frac{1}{N}\left(\frac{\partial T}{\partial n}\right)^2 \frac{\partial \Lambda}{\partial N}.\end{gathered} \tag{1.9}$$

The form of Eq. (1.8) is determined by the number of real roots of the characteristic equation (1.9). When the condition $b^2 + ae > 0$ is satisfied Eq. (1.8) will have a hyperbolic form.

In the simplest case when we have

$$\Lambda = \lambda - \mu \left| \frac{\partial T}{\partial t} \Big/ \frac{\partial T}{\partial x} \right|,\ U = \left(c_0 + c_1 \frac{\partial T}{\partial t} \right) T, \tag{1.10}$$

where λ, μ, c_0, and c_1 are constants of the material, the heat-conduction equation for a one-dimensional rod takes the form

$$\lambda \frac{\partial^2 T}{\partial x^2} - \mu \frac{\partial^2 T}{\partial x \partial t} \operatorname{sign}\left(\frac{\partial T}{\partial t} \frac{\partial T}{\partial x}\right) + \varepsilon = \left(c_0 + c_1 \frac{\partial T}{\partial t}\right) \frac{\partial T}{\partial t} + c_1 T \frac{\partial^2 T}{\partial t^2}. \tag{1.11}$$

We obtain the following value for the velocities of propagation of thermal perturbations:

$$G = \frac{-\mu^* \pm \sqrt{\mu^2 + 4\lambda c_1 T}}{2c_1 T},\ \mu^* = -\mu \operatorname{sign}\left(\frac{\partial T}{\partial t} \frac{\partial T}{\partial x}\right). \tag{1.12}$$

Equation (1.11) can be linearized. To do this we will make the substitution

$$T = T_0 + u, \tag{1.13}$$

where T_0 is the average temperature of the rod at the initial instant of time. In the case when the temperature u varies only slightly with time, and its deviations from T_0 are small, i.e., when

$$|u| \ll T_0,\ c_1\,|\partial u/\partial t| \ll c_0, \tag{1.14}$$

Eq. (1.11), neglecting small terms, takes the form

$$\lambda\,\frac{\partial^2 u}{\partial x^2} + \mu^*\,\frac{\partial^2 u}{\partial x \partial t} + \varepsilon = c_0\,\frac{\partial u}{\partial t} + c_1 T_0\,\frac{\partial^2 u}{\partial t^2}\,. \tag{1.15}$$

When $\mu = 0$ this equation agrees with the heat-conduction equation derived in [1-4]. The difference is solely in the boundary conditions, since the heat flows in [1-4] and in the present paper are defined differently.

2. We will consider whether it is possible for a surface of second-order discontinuity to exist, on which the temperature undergoes an abrupt change. To do this we will introduce a transition layer of thickness 2h, and we will write the equation of conservation of energy (1.1) inside the transition layer in the form [8]

$$G_1 U + \Lambda \partial T/\partial \nu = G_1 U^+ + \Lambda^+ (\partial T/\partial \nu)^+. \tag{2.1}$$

The superscript plus or minus signs denote the value of these quantities in front of or behind the discontinuity surface respectively, G_1 is the velocity of motion of the surface of second-order discontinuity, and ν is the normal to this surface. We will integrate Eq. (2.1) across the transition layer from the back to the front

$$G_1 \int_{-h}^{h} U d\nu + \int_{-h}^{h} \Lambda\,\frac{\partial T}{\partial \nu}\,d\nu = \int_{-h}^{h} \left(G_1 U + \Lambda\,\frac{\partial T}{d\nu}\right)^+ d\nu. \tag{2.2}$$

The functions U and Λ , according to our previous assumptions, are always bounded, so that Eq. (2.2) in the limit as $h \to 0$ takes the form

$$\Lambda^* [T] = 0,\ [T] = T^+ - T^-, \tag{2.3}$$

where Λ^* is the average value of the thermal conductivity inside the transition layer. In the general case $\Lambda^* \neq 0$, and we obtain from Eq. (2.3) that the temperature of the surface of second-order discontinuity is continuous

$$[T] = 0. \tag{2.4}$$

Hence, on a surface of second-order discontinuity only the first and higher partial derivatives of the temperature with respect to the coordinates and time will undergo an abrupt change. To find the rate of propagation of the surface of second-order discontinuity we will write the law of conservation of energy on this surface in the form [8]

$$G\,[U] = -\,[\Lambda \partial T/\partial \nu]. \tag{2.5}$$

When heat propagates in the solid we may encounter the case when a particular isothermal surface exists, which is at the same time a surface of second-order discontinuity. We will consider an example in which such a situation occurs. Consider a solid uniformly heated up to a temperature T_0. Suppose that at the initial instant of time when t = 0 a point source of heat begins to act at a certain point in the solid. For $t > 0$ part of the solid becomes heated. The heated region will form a certain surface Σ which is isolated from the region with undisturbed initial temperature. The surface Σ will be isothermal, on which the temperature is the initial value T_0, and in addition, in general, on this surface the first partial derivatives of the temperature will undergo a discontinuity, i.e., Σ is simultaneously a surface of second-order discontinuity. By definition, the following equations hold on the surface Σ:

$$\left(\frac{\partial T}{\partial t}\right)^{+} = 0, \quad \left(\frac{\partial T}{\partial v}\right)^{-} = \pm N^{-}, \; N^{+} = 0, \; |G_{T}^{-}| = G_1. \tag{2.6}$$

It is seen from Eqs. (2.6) that the velocity G_T^{+} cannot be found from the equation $G_T^{+} = (\partial T/\partial t)^{+}/N^{+}$, and it is therefore best to find it from the equal velocity of the isotherm G_T^{-}

$$G_T^{-} = G_T^{+} = G_T^{*}, \; |G_T^{*}| = G_1. \tag{2.7}$$

Equation (2.5) can be applied to the surface Σ, taking Eqs. (2.6) and (2.7) into account if written in the form

$$G_1 U(T_0, N^{-}, G_T^{*}) + \Lambda(T_0, N^{-}, G_1)(\partial T/\partial v)^{-} = G_1 U(T_0, 0, G_T^{*}). \tag{2.8}$$

Using the additional relations

$$G_1 = G_T^{*} \operatorname{sign}\left(\frac{\partial T}{\partial t}\right)^{-}, \quad \left(\frac{\partial T}{\partial v}\right)^{-} = -N^{-} \operatorname{sign}\left(\frac{\partial T}{\partial t}\right)^{-}, \tag{2.9}$$

Eq. (2.8) takes the form

$$G_T^{*}\{U(T_0, N^{-}, G_T^{*}) - U(T_0, 0, G_T^{*})\} = \Lambda(T_0, N^{-}, G_1) N^{-}. \tag{2.10}$$

When the internal energy U depends only on T and G_T, and is independent of N, we obtain from Eq. (2.10) that there is no heat flow through the surface

$$\Lambda(T_0, N^{-}, G_1) N^{-} = 0. \tag{2.11}$$

Equation (2.11) has two solutions

$$\Lambda^{-} = 0, \text{ for } N^{-} = 0. \tag{2.12}$$

We will consider the second solution $N^{-} = 0$. Then the requirement that the velocity of propagation of the surface Σ is limited implies

$$(\partial T/\partial t)^{-} = 0 \text{ for } N^{-} = 0. \tag{2.13}$$

Hence, the solution $N^{-} = 0$ leads to the condition of continuity of Σ of all the first partial derivatives of the temperature, i.e., Σ will be simultaneously an isothermal and a characteristic surface. Assuming that U is independent of N, from the equation $G = G_T$, where G_T is the velocity of the isotherm, using Eq. (2.13) we obtain the equation

$$\Lambda + N\partial\Lambda/\partial N = 0. \tag{2.14}$$

The function Λ is assumed to be analytic everywhere, so that as $N \to 0$ from Eq. (2.14) we arrive at the first solution in Eq. (2.12).

The above analysis enables us to conclude that the solution $N^-=0$ is a special case of the first solution of Eq. (2.12). We will therefore assume that the solution of (2.11) is

$$\Lambda(T_0,\ N^-,\ G_1)=0. \tag{2.15}$$

Equation (2.15) plays the part of the boundary condition, from which we find G_1 and consequently the position of the surface Σ.

It is seen from Eq. (2.15) that the velocity of the surface Σ, generally speaking, is a variable quantity and depends on the initial temperature and temperature gradient of the isotherm. If Λ is independent of N or T, or of both N and T, the velocity G_1 will be independent of the corresponding quantities. This has important consequences in the experimental determination of the dependence of the heat conduction function on its arguments. In the special case when Λ depends only on $|G_T|$, the velocity G_1 is constant, and is a new thermophysical characteristic.

NOTATION

U, internal energy; Λ, heat conduction function; σ, entropy production velocity; T, absolute temperature; q, heat flux vector; ε, internal sources of thermal energy; t, time; N, temperature gradient modulus; G, characteristic propagation velocity; G_1, propagation velocity of a strong discontinuity surface; G_T, propagation velocity of isothermal surface; λ, μ, c_0, c_1, thermophysical constants of material.

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